Geometry February 2025

Mid-Semester Exam 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{R}$ = real numbers.

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1. [5+7+6 = 18 Points] Let $\mathcal{E}, \mathcal{F}$ be affine spaces over a field k with directions E, F respectively. Let $\varphi\colon \mathcal{E} \to \mathcal{F}$ be a function.
   (i) Define what it means for φ to be an affine mapping in terms of the associated function $\vec{\varphi}\colon E \to F$.
   (ii) Explain why if φ is affine, then the definition of $\vec{\varphi}$ is independent of the choice of a base-point from $\mathcal{E}$.
   (iii) Show that for affine maps, the association $\varphi \mapsto \vec{\varphi}$ respects compositions.

2. [18 Points] Let $\{(A_1, \alpha_1), \ldots, (A_n, \alpha_n)\}$ be a set of weighted points in an affine space $\mathcal{E}$ over a field k. (Thus, $A_i \in \mathcal{E}$ and $\alpha_i \in k$.) Suppose $\sum \alpha_i \neq 0$. Define what it means for a point $G \in \mathcal{E}$ to be a barycentre of this set. Explain why it always exists and why it is unique.

3. [8+16 = 24 Points]
   (i) Let $\mathcal{F}, \mathcal{G}$ be affine subspaces of an affine space $\mathcal{E}$ such that the corresponding directions satisfy $F \oplus G = E$. Prove that $\mathcal{F} \cap \mathcal{G}$ is non-empty and is necessarily a point.
   (ii) With assumptions as in (i), prove that there exists an affine map $\pi\colon \mathcal{E} \to \mathcal{E}$ such that for any point $M \in \mathcal{E}$, we have $\pi(M) \in \mathcal{G}$ and $\overrightarrow{M\pi(M)} \in F$. Moreover, prove that $\pi(M)$ is the intersection $\mathcal{F}' \cap \mathcal{G}$ where $\mathcal{F}' \subset \mathcal{E}$ is the unique affine space with direction F passing through M.

4. [6+6+6 = 18 Points] In each of the following cases below, find an example of a Euclidean affine space $\mathcal{E}$, a point $O \in \mathcal{E}$ and an affine bijection $\varphi\colon \mathcal{E} \to \mathcal{E}$ satisfying the given property.
   (i) $\varphi(O) = O$ and φ preserves oriented angles between segments through O, but φ does not preserve lengths of the segments.
   (ii) φ is an isometry with $\varphi(O) = O$ but φ does not preserve the oriented angles between segments through O.
   (iii) φ is an isometry but has no fixed points.

5. [8 +14 = 22 Points] Let $\mathcal{E}$ be a Euclidean affine space of dimension $n \geq 1$ and let $\varphi\colon \mathcal{E} \to \mathcal{E}$ be an affine map.

(i) Prove that the set of all fixed points of φ is an affine subspace $\mathcal{F}$ of $\mathcal{E}$.

(ii) If φ is an isometry and $\mathcal{F}$ from (i) is non-empty with $\dim(\mathcal{F}) = d$, then prove that φ is a composition of at most $n - d$ reflections.

(Note: You may assume the existence and equidistance property of a perpendicular bisector hyperplane for any closed segment $\overline{AB}$ in $\mathcal{E}$ where $A \neq B$.)